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Journal of Banking & Finance 28 (2004) 997–1021
Journal of BANKING & FINANCE
www.elsevier.com/locate/econbase

Credit derivatives with multiple debt issues ☆

Pascal François [a,1], Georges Hübner [b,c,*]

[a] HEC Montreal, Finance Department, 3000 Cote-Ste-Catherine, H3T 2A7 Montreal, Canada
[b] Dept. of Economics, Business Administration and Social Sciences, Service of Financial Management, Université de Liège, Boulevard du Rectorat 7, Bat. B31, B-4000 Liège, Belgium
[c] Limburg Institute of Financial Economics, Faculty of Economics and Business Administration, Maastricht University, Postbus 616, NL-6200 MD, Maastricht, The Netherlands

Received 19 July 2002; accepted 2 January 2003

Abstract

We evaluate the most actively traded types of credit derivatives within a unified pricing framework that allows for multiple debt issues. Since firms default on all of their obligations, total debt is instrumental in the likelihood of default and therefore in credit derivatives valuation. We use a single factor interest rate model where the exponential default frontier is based on total debt and is made coherent with observed bond prices. Analytical formulae are derived for credit default swaps, total return swaps (both fixed-for-fixed and fixed-for-floating), and credit risk options (CROs). Price behaviors and hedging properties of all these credit derivatives are investigated. Simulations document that credit derivatives prices may be significantly affected by terms of debt other than those of the reference obligation. The analysis of CROs indicates their superior ability to fine-tune the hedging of magnitude and arrival risks of default.

JEL classification: G12; G13
Keywords: Credit derivatives; Structural models; Debt structure; Options; Swaps

☆ Pascal François acknowledges financial support from SSHRC, IFM2 and HEC Montréal. Georges Hübner acknowledges financial support from Deloitte and Touche Luxemburg. Part of this research was completed while the second author was visiting HEC Montréal. We thank Georges Dionne, Geneviève Gauthier, participants of the 2001 Southern Finance Association meetings as well as two anonymous referees for helpful comments.
* Corresponding author. Tel.: +32-43662765; fax: +32-43662821.
E-mail addresses: pascal.francois@hec.ca (P. François), g.hubner@ulg.ac.be (G. Hübner).
[1] Tel.: +1-514-340-7743; fax: +1-514-340-5632.

0378-4266/$ - see front matter
doi:10.1016/S0378-4266(03)00048-7

Credit derivatives are financial contracts that allow one party to transfer credit risk to another party. They permit the trading of credit risk separately from other sources of risk. The market for credit derivatives, with New York and London being its most active places, is estimated between $400 billion and $1 trillion notional outstanding according to Hargreaves (2000). Its exponential growth within the last ten years has retained the attention of many researchers in credit risk pricing. Contributions are commonly classified the same way as the literature on corporate bond pricing.

"Reduced form" models (or "intensity" models) directly specify the dynamics of the bond price and view default as an unpredictable event (e.g. the first jump of a Poisson process). This approach, initiated by Jarrow and Turnbull (1995), has been extended to credit derivatives by Flesaker et al. (1994), Kijima and Komoribayashi (1998), Duffie and Singleton (1999), or Chen and Sopranzetti (2002).

By contrast, "structural" models (or "firm value" models) specify the dynamics of the firm assets value and use contingent claims analysis to price the securities issued by this firm. This is the approach we follow here. The seminal reference is Merton (1974) yielding a solution to the price of a corporate discount bond. Extensions of this work evaluate more sophisticated debt contracts (see e.g. Black and Cox (1976) for debt with a safety covenant; Geske (1977) for coupon-bearing debt, or Brennan and Schwartz (1980) for convertible debt). Other extensions evaluate debt contracts under more complex default rules (see e.g. Leland (1994) where the decision to default is driven by the capital structure static trade-off; Anderson and Sundaresan (1996) where debt is subject to strategic default, or François and Morellec (2002) where default leads to a Court-supervised reorganization procedure). Applications of structural models to credit derivatives include the works of Longstaff and Schwartz (1995a), Das (1995), Pierides (1997) and Ammann (2001). All these papers mostly study credit risk options (referred to as CROs). The model of Longstaff and Schwartz (1995a) relies on the empirically supported assumption that credit spreads are mean-reverting. Their model values European CROs that pay the credit spread of a bond at a given date. However, this pay-off is not triggered by the event of default, and the option is therefore inadequate for hedging purposes. Das (1995) derives a closed-form formula in continuous time for the value of a CRO written on a bond paying a continuous coupon. Interest rate is assumed constant. He further develops a discrete time model with a Heath–Jarrow–Morton term structure model, but no analytical result is available. Das and Sundaram (2000) work in that same framework, but directly model the credit spread. Hence they value the same kind of CROs as in Longstaff and Schwartz (1995a). Ammann (2001) adopts a compound option approach to value CROs written on discount bonds. In these three papers, default is exogenous. By contrast, Pierides (1997) values CROs with an endogenous default rule at the cost of assuming that interest rates are constant. Recently, Bélanger et al. (2001) have developed a general pricing framework for credit derivatives that embed both structural and reduced form models.

With respect to the literature on contingent claims models for credit derivatives, our contribution consists of four points. First, we provide closed form formulae not only for CROs but also for all other most actively traded credit derivatives within a

unified pricing framework. Specifically, we study credit default swaps (CDS), fixed-for-fixed total return swaps (fixed TRS), and fixed-for-floating TRS (floating TRS), and we make a distinction between default CRO and downgrading CRO which respectively compensate for the total default loss or for a pre-specified downgrading of the reference obligation. Second, consistent with CROs as hedging instruments, our model makes their pay-off contingent upon the default or more generally upon the downgrading event. Third, our model incorporates stochastic interest rates and a default policy that is made coherent with observed bond prices. Fourth, our model makes a clear distinction between the reference obligation underlying the credit derivative and the issuer's total debt. The issuer is allowed to have multiple classes of debt outstanding with different nominal rates, maturities and seniorities. This specification enables to capture the whole information contained in the debt structure about the default likelihood. In particular, it measures the impact of a marginal change in the debt structure on credit derivative pricing.

In line with "structural models", we characterize the event of default using the first passage time approach. This default modeling is pioneered by Black and Cox (1976) where risk free rate is assumed constant. Extensions are provided in two notable directions. On one hand, Kim et al. (1993) or Longstaff and Schwartz (1995b) introduce interest rate risk but use constant barriers. Though a constant barrier may be justified by a particular debt provision (such as a solvency constraint or a safety covenant), the general case is that shareholders have the unrestricted (American) option to default, which implies a default boundary that is better approximated by an exponential function for finitely lived debt. On the other hand, some papers like Saà-Requejo and Santa-Clara (1999) assume a stochastic default boundary. Though more comprehensive, this setting complicates model calibration by introducing additional parameters. Moreover, it generally admits no analytical solution and requires the use of numerical methods. An exception is Finkelstein et al. (2002) who obtain a closed-form formula for CDS at the cost of assuming a constant risk free rate. We adopt the in-between approach followed by Briys and de Varenne (1997) where the exogenous default boundary is related to default-free bond prices. In extension of their work however, we explicitly link the boundary to the firm capital structure. This view is consistent with a shareholders' decision to default based on total debt.

Our main findings may be classified in two categories, one regarding hedging properties of credit derivatives and the other regarding term structures of credit derivatives premia. Hedging properties of CDS and fixed TRS are very similar. Interest rate risk as well as credit risk hedging performance of TRS vary with the issuer's leverage. CROs are the most effective credit risk hedging tools. By contrast with other credit derivatives, their maturity date is an additional degree of freedom that allows to fine-tune the hedge between credit risk intensity and magnitude. Term structures of CDS and TRS premia are similar in shape as those of corporate yield spreads. Term structures of CRO premia exhibit a bell shape. All term structures are affected in a non-trivial way by marginal changes in the issuer's debt structure. Section 1 presents the pricing framework. Section 2 derives pricing formulae for credit derivatives and conducts simulations analysis. Section 3 concludes. Technical proofs are gathered in Appendix A.

1. The pricing framework

In this section, we present and discuss the seven assumptions of the model. The corporate discount bond term structure is similar to the Briys and de Varenne (1997) setting, and we derive the formula for the coupon-bearing reference obligation as a direct extension of their work.

Assumption 1. Assets are continuously traded in arbitrage-free and complete markets.

Assumption 2. The reference obligation is issued by a firm whose assets value, denoted V, follows a lognormal diffusion under the risk neutral probability measure $\mathbb{Q}$, i.e.,

$$\frac{\mathrm{d}V_t}{V_t} = (r_t - \delta)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

where σ is the constant volatility of the returns on the firm's assets, $(r_t, t \geqslant 0)$ is the stochastic process representing the instantaneous riskfree rate, δ is the constant payout rate, and $(W_t, t \geqslant 0)$ is a standard Brownian motion accounting for the business risk of the firm. From Assumption 1 and Harrison and Pliska (1981), we know $\mathbb{Q}$ is unique.

Assumption 3. To finance its assets, the firm has equity outstanding and K debt issues with face value F_j paying coupons c_j and maturity T_j for $j = 1, \ldots, K$. We rank these issues by increasing maturities $T_1 \leqslant \cdots \leqslant T_K$. One of these debt issues is the reference obligation underlying the credit derivatives under study. It has face value F, maturity T, and pays n coupons c at dates t_i, $i = 1, \ldots, n$. Its current market price is denoted $P_V(0, c, F, T)$. For simplicity, we assume the instantaneous cash outflow is the sum of coupon payments $((1/V_t)\sum c_i)$ and dividend payments which are adjusted so as to keep the payout rate constant. [2]

Let T^* represent stockholders' horizon of decision (e.g. the firm's predicted end of operations and liquidation of its assets). We only require that $T^* \gg T_K$ to ensure that the shape of the default threshold remains the same during the life of the reference obligation. The next assumption characterizes the default policy adopted by stockholders.

Assumption 4. The default boundary H_t is given by

$$H_t = \lambda M_t P(t, T^*)$$

[2] This specification is consistent with typical debt covenants that impose a ceiling on the firm cash outflows (see Smith and Warner, 1979).

with

$$M_t = \sum_{j=1}^{k(t)} \frac{F_j B(T_j, t)}{P(t, T^*)} + \sum_{j=k(t)+1}^{K} \frac{F_j}{P_{f,t}(T_j, T^*)}$$

and

$$k(t) = \sup_j (T_j \leqslant t),$$

where λ is a constant, $P(t,u)$ stands for the time-t discount factor for maturity $u \geqslant t$, $B(s,t) = P(t,u)/P(s,u)$ represents the time-t value of 1 dollar invested on the money market account at time $s \leqslant t$, and $P_{f,s}(t,u)$ is the time-s forward price of the discount factor from maturity u to date t.

Assumption 4 deserves the following comments. First, the value M_t represents the stochastic total stock of debt at time t to be paid at the stockholders' horizon. It is calculated as the total principal amount of debt capitalized from t to T^*. This reflects that stockholders assess the decision to default on the basis of total debt. It is made up with two components: (i) principals maturing before t are capitalized from T_j to t using the money market account (known at date t) and then from t to T^*, and (ii) principals maturing after t are capitalized until T^* using the forward discount factor computed at date t. Second, in the spirit of Black and Cox (1976) and Briys and de Varenne (1997), the default threshold is continuous with an exponential shape. Default may therefore occur at any date within stockholders' horizon of decision T^*. Parameter λ is the strategic instrument conditioning the likelihood of default. Third, coupons do not impact on the default frontier. It is consistent with a large body of literature that considers default under stock-based conditions. [3]

Lemma 1. *Under Assumption 4, the timing of default does not depend on the stockholders' horizon of decision, and is represented by the stopping time θ defined by*

$$\theta = \inf \left\{ t \geqslant 0 : V_t = \lambda \frac{\Phi}{P(0,T)} P(t,T) \right\},$$

where $\Phi = \sum_{j=1}^{K} F_j P(0, T_j)$ stands for the current total amount of debt.

Proof. See Appendix A.

Assumption 5. Every debtholder recovers a fraction Γ_j of their discounted nominal, and therefore gets $\Gamma_j \lambda F_j P(\theta, T_j)$ upon default. In particular, the reference obligation recovery rate is denoted by $\Gamma\lambda$. This assumption is equivalent to the recovery of treasury value paid *at the default date* (see e.g. Briys and de Varenne, 1997; Duffie and Singleton, 1999). Our specification has two appealing features. First, it is equivalent to the recovery of nominal value paid at maturity, which cancels out a stochastic discount factor for debt recovery value. Second, for empirical application, recovery rates expressed as a fraction of nominal value (the product $\Gamma_j \lambda$ in our

[3] In frictionless markets, stock-based and flow-based definitions of default are equivalent. See Nielsen et al. (1993) for a discussion.

model) are observable and may be calibrated from the issue's degree of priority (see e.g. the extensive survey of Altman and Kishore (1996) on observed recovery rates).

At this point, it seems useful to discuss the respective roles of parameters λ and Γ in our modeling. Parameter λ represents the critical value of the ratio (assets value)/(total amount of debt) below which shareholders decide to leave the firm to creditors. Thus, λ may be interpreted as the debt recovery rate for debtholders *as a whole* (i.e. considered as a homogeneous class of claimants). However, since debt issues may differ in seniority, and also since third-party costs may apply upon default, the effective recovery rate for *each* debt issue is adjusted by the idiosyncratic factor Γ_j. For model implementation, parameter λ is unobservable but may be inferred from the price of the underlying reference obligation (using Eq. (1) below). By contrast, the product $\Gamma\lambda$ may be calibrated using statistical data on recovery rates.

Assumption 6. The instantaneous riskless interest rate follows an Ornstein–Uhlenbeck process, that is

$$\mathrm{d}r_t = \alpha(\beta - r_t)\mathrm{d}t + \sigma_\mathrm{r}\mathrm{d}Z_t,$$

where β represents the long-term equilibrium value of the process, α is its mean reversion speed, and σ_r is the interest rate volatility. All three parameters are assumed constant. [4] Moreover, the process $(Z_t, t \geqslant 0)$ is another standard Brownian motion and represents interest rate uncertainty. We denote by ρ the correlation coefficient between $(W_t, t \geqslant 0)$ and $(Z_t, t \geqslant 0)$. The well-known expression of $P(t,T)$ is given by Vasicek (1977):

$$P(0,t) = a(t)\exp(-b(t)r)$$

with

$$a(t) = \exp\left(\frac{(b(t)-t)(\alpha^2\beta - \sigma_\mathrm{r}^2/2)}{\alpha^2} - \frac{\sigma_\mathrm{r}^2 b^2(t)}{4\alpha}\right),$$

$$b(t) = \frac{1-\exp(-\alpha t)}{\alpha}.$$

Within this pricing framework, Briys and de Varenne (1997) provide the closed-form solution for the value of a risky discount bond. Their result can be readily extended to a reference bond with coupons, which yields

$$\begin{aligned} P_V(0,c,F,T) &= FP(0,T)\left(\mathcal{N}(-d_1) - \frac{V}{\lambda\Phi}\mathcal{N}(d_2)\right) + \Gamma\lambda FP(0,T)\left(\mathcal{N}(d_1) + \frac{V}{\lambda\Phi}\mathcal{N}(d_2)\right) \\ &\quad + \sum_{i=1}^{n} cP(0,t_i)\left(\mathcal{N}(-d_1^i) - \frac{V}{\lambda\Phi}\mathcal{N}(d_2^i)\right) \end{aligned} \tag{1}$$

[4] The framework can be extended to a generalized Vasicek model where the long-term equilibrium value β is made time-dependent to capture the information contained in the initial term structure (see Hull and White (1990) for a description). When implementing a credit derivative valuation model, this extension helps prevent arbitrage opportunities. For clarity of exposure, we restrict the analysis to the standard Vasicek model. This restriction however does not alter qualitatively the results of our simulations.

with

$$d_1 = \frac{1}{s(0,T)}\left(\ln\frac{\lambda\Phi}{V} + \delta T + \frac{s^2(0,T)}{2}\right), \quad d_2 = d_1 - s(0,T),$$

$$d_1^i = \frac{1}{s(0,t_i)}\left(\ln\frac{\lambda\Phi}{V} + \delta t_i + \frac{s^2(0,t_i)}{2}\right), \quad d_2^i = d_1^i - s(0,t_i),$$

$$s(0,v) = \sqrt{\int_0^v [(\rho\sigma + \sigma_p(u,v))^2 + (1-\rho^2)\sigma^2]\mathrm{d}u},$$

$$\sigma_p(u,v) = \frac{\sigma_r}{\alpha} \times (1 - \mathrm{e}^{-\alpha(v-u)}),$$

where $\mathcal{N}(\cdot)$ denotes the cumulative normal distribution function, and $\sigma_p(u,v)$ is the volatility of the riskless discount bond.

Assumption 7. Credit derivatives are not subject to credit risk themselves.

This simplification may be justified by market practice. CROs being hedging instruments, their issuers are typically triple A-rated banks. Accordingly, we consider these derivatives exempt of default risk. Hübner (2001) evaluates interest rates and currency swaps where both parties are subject to credit risk.

2. Pricing credit derivatives

Credit derivatives are OTC products and a wide variety of contract design is available. A detailed presentation of products and market practices may be found in Das (1998) or in Tavakoli (1998). The following credit derivatives typology is based on their classifications.

(1) CDS: A pays B a fixed periodic amount, while B pays A an agreed amount upon default of the reference obligation. Usually, this amount is the loss in market

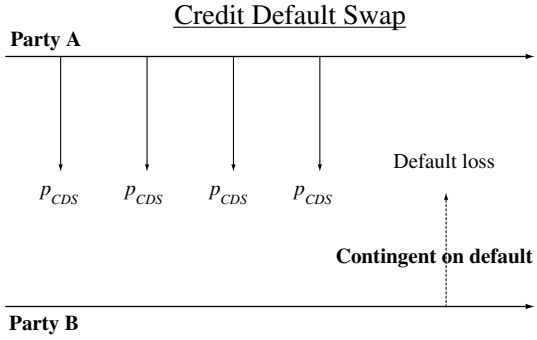

Fig. 1. The payoffs of the CDS.

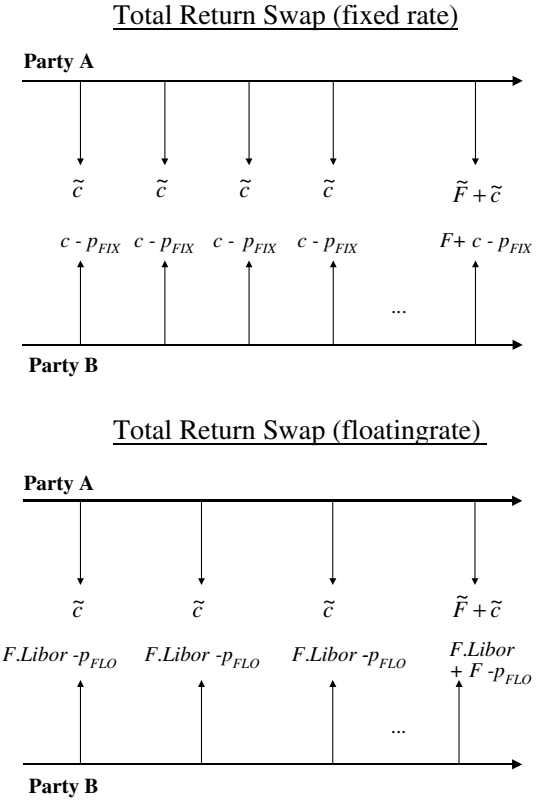

Fig. 2. The payoffs of the TRS. Notation $\tilde{c}$ means that the coupon payment is conditional on the bond survival. Notation $\widetilde{F}$ corresponds to the face value if the bond has not defaulted, and to the capitalized default payoff $\Gamma\lambda F$ otherwise.

value due to default. The payoffs involved by a CDS can be represented as follows (Fig. 1).

(2) TRS: On a periodic basis, A pays B interest payments made on the principal amount of the reference obligation, while B pays A a fixed return or a floating return (e.g. Libor) on that principal amount. Default usually does not lead to contract termination, and B commits to paying until debt maturity. The payoffs involved by a TRS can be represented as follows (Fig. 2).

(3) CRO: A pays B a fixed amount today, while B compensates A for the loss in market value of the reference obligation due to a pre-specified fall in credit standing (not necessarily default). The pre-specified fall is the strike of the CRO. The payoffs involved by a CRO can be represented as follows (Fig. 3).

A survey conducted on the London market and quoted by Ammann (2001) reports that credit derivatives 1996 market shares were as follows: CDS 35%, TRS 17%, CRO 15% and others (hybrids) 33%.

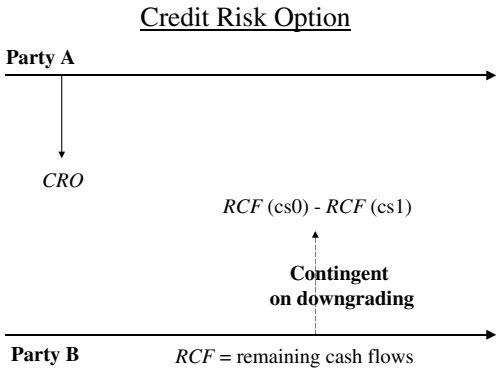

Fig. 3. The payoffs of the CRO. RCF(cs$x$) stands for the remaining cash flows discounted at rate cs$x$.

2.1. Credit default and total return swaps

In this section, we first price CDS and TRS in the absence of arbitrage. Then, using simulations, we investigate the hedging properties and the term structures of these derivatives. The following two propositions give CDS and TRS premia.

Proposition 2. *Under Assumptions 1–7, the premium of the* CDS, *denoted by* p_{cds}, *satisfies*

$$p_{\text{cds}} = \frac{c\sum_{i=1}^{n} P(0,t_i) + FP(0,T) - P_V(0,c,F,T)}{\sum_{i=1}^{n} P(0,t_i)[\mathcal{N}(-d_1^i) - \frac{V}{\lambda\Phi}\mathcal{N}(d_2^i)]} \tag{2}$$

with $P_V(0,c,F,T)$ *is the current value of the reference obligation given by Eq.* (1).

Proof. See Appendix A.

Proposition 3. *Under Assumptions 1–7, the premia* p_{fix} *and* p_{flo} *of the fixed* TRS *and of the floating* TRS *respectively satisfy*

$$p_{\text{fix}} = c + \frac{FP(0,T) - P_V(0,c,F,T)}{\sum_{i=1}^{n} P(0,t_i)}, \tag{3}$$

$$p_{\text{flo}} = \frac{\sum_{i=1}^{n} FP(0,t_i)y(t_i,t_{i+1}) + FP(0,T) - P_V(0,c,F,T)}{\sum_{i=1}^{n} P(0,t_i)}, \tag{4}$$

where

$$y(t_i,t_{i+1}) = \frac{-\ln a(t_{i+1}-t_i)}{t_{i+1}-t_i} + \frac{b(t_{i+1}-t_i)}{t_{i+1}-t_i}\left((r-\beta)\mathrm{e}^{-\alpha t_i} + \beta - \frac{\sigma_r^2}{2\alpha^2}(1-\mathrm{e}^{-\alpha t_i})^2\right).$$

Proof. See Appendix A. One can check that fixed TRS and floating TRS premia are time-varying affine transformations of CDS premia.

2.1.1. CDS and TRS hedging properties

For simulation purposes, our base case parametrization is the following:

β	r	σ_r	α	V	σ	ρ	Γ	λ	δ
0.06	0.05	0.02	0.1	1000	0.2	−0.25	0.6	0.6	0.03

In our model, the product $\Gamma\lambda$ represents the recovery rate of the reference obligation. On a sample of 696 bonds that defaulted between 1978 and 1995, Altman and Kishore (1996) report average recovery rates ranging from 21% to 58% depending on the seniority and the collateralization of the issue. Values of α and σ_r are based on empirical tests of the Vasicek model. Estimation of this model by Sanders and Unal (1988) on US data reports speed reversion values ranging from 0.025 to 0.248, and volatility coefficients ranging from 0.022 to 0.04. Studies by de Munnik and Schotman (1994) and Sercu and Wu (1997) on respectively Dutch and Belgian data find a mean speed reversion of 0.116 and 0.101, and a mean volatility of 0.0284 and 0.0277 respectively. Values of β and r reflect the current term structure of interest rates with instantaneous riskless rate at 5% and an upward sloping curve. [5] The other parameters correspond to a standard calibration.

We report in Table 1 the comparative statics of fixed TRS and floating TRS premia. CDS premia exhibit parameter sensitivities that are extremely close to those of fixed TRS premia and are therefore not reported. Parameters are classified in three categories. First, we have separated interest rate parameters from firm characteristics as fixed and floating TRS premia react in drastically different ways. Then, among firm specific inputs, default threshold λ and payout rate δ can be viewed as strategic variables in that they reflect shareholders' financing decisions.

Premia sensitivities are listed for three levels of quasi-debt ratio (QDR) defined as the sum of discounted (at the riskless rate) principals plus coupons over initial assets value. As shown by Merton (1974), QDR is a reliable proxy for corporate leverage.

The fixed TRS is best suited for hedging against default risk, especially for low leverages: at a QDR of 40%, it exhibits sensitivities with respect to firm specific parameters that are consistently around 14 times that of floating TRS. On the other hand, the floating TRS offers a significant hedge against interest rate movements at the expense of a lower credit risk hedging effectiveness. For high leverage ratios however, the floating TRS gets almost as efficient as the fixed TRS in terms

[5] Simulations with a downward sloping yield curve display qualitatively similar results. More specifically, two effects are at stake. First, as the riskless interest rate declines, so does the drift of the state variable and therefore the default probability increases, which raises the premia of all credit derivatives. Second, credit derivatives promising fixed payoffs (such as CDS or fixed TRS) have their future discounted payoff raised. For these derivatives therefore, the premia significantly increase when the yield curve is downward sloping compared to the upward sloping case. However, for credit derivatives promising interest-rate-linked payoffs (such as floating TRS or CRO), the discount effect partially offsets the default probability effect.

Table 1
Comparative statics of TRS premia

Initial value		QDR = 40%		QDR = 60%		QDR = 80%	
		p^{-}_{fix} 0.0197	p_{flo} 0.2830	p_{fix} 0.7619	p_{flo} 1.0251	p_{fix} 3.9138	p_{flo} 4.1771
Interest rate environment	β	+1.86	+131.26	+0.40	+36.50	−0.03	+8.86
	β, r	+7.64	+375.78	+2.04	+105.10	+0.41	+25.80
	σ_r	+2.41	−26.25	+1.18	−6.41	+0.63	−1.20
	α	−0.26	+22.56	−0.20	+6.09	−0.14	+1.40
Firm characteristics	V	−58.00	−4.03	−45.58	−33.87	−36.60	−34.29
	σ	+220.14	+15.31	+72.84	+54.14	+32.78	+30.71
	ρ	−8.47	−0.59	−4.02	−2.98	−2.05	−1.92
	Γ	−5.13	−0.36	−5.00	−3.71	−4.81	−4.51
Strategic variables	λ	+89.78	+6.24	+52.89	+39.31	+35.02	+32.81
	δ	−25.23	−1.75	−5.50	−4.08	+0.36	+0.34

Price variations of credit derivatives (in percentage of initial value) are obtained with a 10% variation of each parameter. Initial interest rate environment is characterized by: mean reversion coefficient $\alpha = 0.1$, long term mean of riskless rate $\beta = 0.06$, current spot rate $r = 0.05$, and interest rate volatility $\sigma_r = 0.02$. Initial firm characteristics are: current assets value $V = 1000$, principals of debt outstanding $F_1 = F_2 = 200$, and F_3 is such that the firm QDR remains constant, debt maturities $T_1 = 2$, $T_2 = 6$, and $T_3 = 10$, volatility of firm's assets returns $\sigma = 0.2$, recovery rate $\Gamma = 0.6$, and correlation coefficient between interest rate and firm value process $\rho = -0.25$. Initial strategic variables are: constant payout rate $\delta = 0.03$, and default threshold $\lambda = 0.6$. To generate payout rate variations, coupon payments are initially set at $c_1 = c_2 = c_3 = 10$ and move together by the same proportions. Credit derivatives are written on the reference obligation with price $P_V(0, c_2, F_2, T_2)$.

of credit risk hedging, while its interest rate hedging performance remains significantly better.

Indeed, as leverage increases, default gets more likely, the moneyness of the fixed TRS increases (initial value goes from 0.0197 to 3.9138), while the interest rate hedging component of the floating TRS, which roughly remains constant at 0.26, becomes marginal. Thus, the behavior with respect to firm specific variables (including strategic variables) of the floating TRS eventually tends to mimic the decreasing sensitivities of the fixed TRS.

In addition, fixed and floating TRS premia exhibit sensitivities of the same direction with the notable exception of the two interest rate risk parameters (σ_r and α). Suppose the instantaneous riskless rate is on a rising trend ($\beta > r$), which is our case. Reducing risk to this trend (by lowering σ_r or raising α) adds value to the floating TRS whose payments are based on the riskless rate. On the contrary, it reduces the value of the fixed TRS which promises fixed discounted payments.

Finally, the sensitivities of both TRS premia with respect to δ change signs as leverage increases. This is due to two opposite effects: the duration effect of increasing coupons that reduces the value of the derivatives, and the positive effect of making default more likely by raising δ.

2.1.2. CDS and TRS term structures

We move on to analyzing the term structures of CDS and TRS premia. [6] We only report simulations of term structures of CDS premia since those of TRS premia exhibit similar shapes. In Figs. 4 and 5, we consider a firm with three debt issues: short term debt with a two-year maturity, long term debt with a 10-year maturity, and the credit derivative reference obligation. Leverage is kept constant at 60% across all maturities by controlling the characteristics of the long term debt.

The modelling of credit derivatives with multiple debt issues enables us to carry out an analysis of CDS premia term structures based on three main dimensions:

1. the importance of the reference obligation in the payout ratio δ that penalizes the drift of the firm value process;
2. the importance of the reference obligation in the sum of discounted principals Φ that influences the default threshold;
3. the importance of the reference obligation in the QDR that serves as a proxy for total leverage.

Fig. 4 plots term structures of CDS premia when the proportion of coupons paid on the reference obligation in debt service δ remains constant across each case, while the impacts of this bond on Φ and on the QDR vary.

The graph is constructed so that triangles, squares and crosses are ranked by decreasing levels of Φ and increasing proportion of the reference obligation face value in Φ. The curves never intersect. Indeed, decreasing Φ amounts to reducing the default probability, and consequently the CDS value, too. In addition, CDS term structures are increasing. On one hand, the expected present value of the default loss declines with the CDS maturity, but on the other hand, the probability of exercising the (American) option increases, and this second effect more than offsets the first one (this result is robust across various coupon calibrations).

Fig. 5 plots term structures of CDS premia when the proportion of the reference obligation in the QDR remains constant across each case, while the impact of this bond on δ and on Φ varies.

The graph is constructed so that triangles, squares and crosses are ranked by increasing levels of δ and decreasing levels of Φ. Conversely, the reference obligation has coupons with a decreasing weight in δ, and a face value with increasing weight in Φ. The intersections of the reported term structures suggest a drastic change of the relative impact of these variables for short term and long term bonds. As the long term debt coupon increases, debt service raises the default probability, but also F_3 is reduced (to keep QDR constant), which lowers the default threshold. The first effect dominates on the long run.

In sum, Figs. 4 and 5 therefore suggest that renegotiation of out-of-reference debt contracts causes significant changes in credit derivative prices. Depending on the

[6] The model is able to generate increasing, humped or decreasing term structures for various QDR and default thresholds (not reported), which is a standard result of contingent claims models (see Pitts and Selby, 1983).

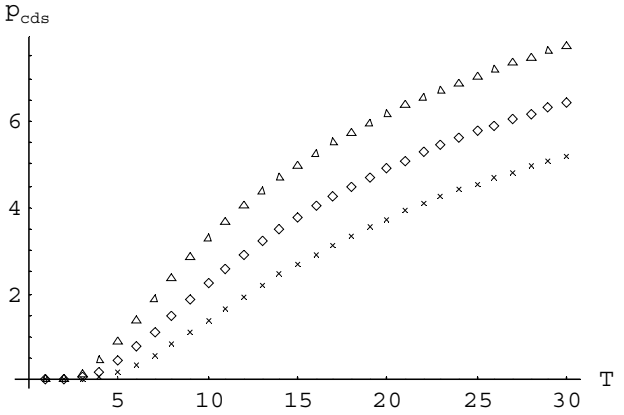

Fig. 4. Term structures of CDS premia for various coupon payment distributions. Reported coupon payment distributions are $c_1 = 20$, $c_2 = 10$, $c_3 = 0$ (triangles), $c_1 = c_2 = c_3 = 10$ (squares), and $c_1 = 0$, $c_2 = 10$, $c_3 = 20$ (crosses). Interest rate environment is characterized by: mean reversion coefficient $\alpha = 0.1$, long term mean of riskless rate $\beta = 0.06$, current spot rate $r = 0.05$, and interest rate volatility $\sigma_r = 0.02$. Firm characteristics are: current assets value $V = 1000$, principals of debt outstanding: $F_1 = F_2 = 200$, and F_3 is such that the firm QDR is constant at 60%, debt maturities: $T_1 = 2$ and $T_3 = 10$, volatility of firm's assets returns $\sigma = 0.2$, recovery rate $\Gamma = 0.6$, default threshold $\lambda = 0.6$, and correlation coefficient between interest rate and firm value process $\rho = -0.25$. The CDS is written on the reference obligation with price $P_V(0, c_2, F_2, T_2)$.

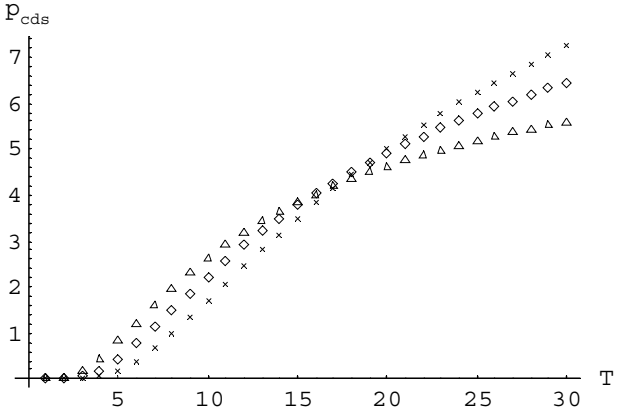

Fig. 5. Term structures of CDS premia for various debt services and total debt. Reported debt services are $c_1 = 10$, $c_2 = 10$, and $c_3 = 0$ (triangles), $c_1 = 10$, $c_2 = 10$, and $c_3 = 10$ (squares), and $c_1 = 10$, $c_2 = 10$, and $c_3 = 20$ (crosses). Interest rate environment is characterized by: mean reversion coefficient $\alpha = 0.1$, long term mean of riskless rate $\beta = 0.06$, current spot rate $r = 0.05$, and interest rate volatility $\sigma_r = 0.02$. Firm characteristics are: current assets value $V = 1000$, principals of debt outstanding: $F_1 = F_2 = 200$, and F_3 is such that the firm QDR is constant at 60%, debt maturities: $T_1 = 2$ and $T_3 = 10$, volatility of firm's assets returns $\sigma = 0.2$, recovery rate $\Gamma = 0.6$, default threshold $\lambda = 0.6$, and correlation coefficient between interest rate and firm value process $\rho = -0.25$. The CDS is written on the reference obligation with price $P_V(0, c_2, F_2, T_2)$.

credit derivative maturity, misspecifying terms of debt other than the reference obligation may lead to substantial upward or downward pricing biases.

2.2. Credit risk options

CROs encompass a vast variety of contract designs. Unlike swaps, the payoff of the CRO is not necessarily contingent on default of the reference bond. Indeed, a downgrading of the bond is sufficient to trigger the exercise of the CRO. Furthermore, the maturity of the option usually differs from the one of the underlying bond. These two differences are likely to bring up several new interpretations, as shown below.

2.2.1. Default CRO

The default CRO is an American option: at any time before the option maturity date t_d, the option is exercised if the underlying bond is defaulted, and the writer of the option must pay the present value of all the coupons and the principal remaining to be paid henceforth. Unlike most types of American options, the default CRO bears an analytical formulation in this framework. This is due to the fact that the strike of the option is automatic, i.e. contingent on the event of default. The pricing of this option is given in the following proposition.

Proposition 4. *Under Assumptions 1–7, the premium* p_{def} *of the default* CRO *maturing at* $t_d < T$ *satisfies*

$$p_{\text{def}} = c\left(\sum_{i=1}^{l} x_i + x_d \sum_{i=l+1}^{n} P_f(t_d, t_i)\right) + (1 - \lambda\Gamma)Fx_d P_f(t_d, T),$$

where

$$x_i = P(0,t_i)\mathcal{N}(d_1^i) + \frac{VP(0,t_i)}{\lambda\Phi}\mathcal{N}(d_2^i),$$

$$l = \sup\{i \geqslant 0 : t_i \leqslant t_d\}$$

and $x_d = x_i$ *where* $t_i = t_d$. $P_f(t_d, t)$ *stands for the forward price of the zero-coupon bond and is given by*

$$P_f(t_d, t) = \exp\left[-\int_{t_d}^{t}\left((r-\beta)e^{-\alpha u} + \beta - \frac{\sigma_r^2}{2\alpha^2}(1-e^{-\alpha u})^2\right)du\right].$$

When $t_d = T$, *the premium becomes*

$$p_{\text{def}} = c\sum_{i=1}^{n} P(0,t_i) + FP(0,T) - P_V(0,c,F,T) = p_{\text{fix}}\sum_{i=1}^{n} P(0,t_i).$$

Proof. See Appendix A.

Proposition 3 displays a clear parallelism between the pricing of a CRO and the fixed-for-fixed swap, with an even linear relationship between their premia when the option and bond maturities coincide. For this optional instrument to bring additional hedging interest with respect to the corresponding swap, it is necessary that

Table 2
Comparative statics of TRS and CRO premia

Initial value		p_{fix} 0.7619	p_{flo} 1.0251	p_{def} (2) 0.0021	p_{def} (4) 0.5623	p_{def} (6) 3.8399
Interest rate environment	β	+0.40	+36.50	+2.17	+0.81	+0.04
	β, r	+2.04	+105.10	+8.27	+2.32	+0.29
	σ_r	+1.18	−6.41	−2.94	−0.47	+1.26
	α	−0.20	+6.09	+0.55	+0.16	−0.26
Firm characteristics	V	−45.58	−33.87	−81.88	−58.88	−45.58
	σ	+72.84	+54.14	+524.61	+140.54	+72.85
	ρ	−4.02	−2.98	−4.95	−4.53	−4.02
	Γ	−5.00	−3.71	−3.92	−4.46	−4.99
Strategic variables	λ	+52.89	+39.31	+329.29	+99.22	+52.90
	δ	−5.50	−4.08	−33.21	−12.99	−5.50

Price variations of credit derivatives (in percentage of initial value) are obtained with a 10% variation of each parameter. Initial interest rate environment is characterized by: mean reversion coefficient $\alpha = 0.1$, long term mean of riskless rate $\beta = 0.06$, current spot rate $r = 0.05$, and interest rate volatility $\sigma_r = 0.02$. Initial firm characteristics are: current assets value $V = 1000$, principals of debt outstanding $F_1 = F_2 = 200$, and F_3 is such that the firm QDR is equal to 60%, debt maturities $T_1 = 2$, $T_2 = 6$, and $T_3 = 10$, volatility of firm's assets returns $\sigma = 0.2$, recovery rate $\Gamma = 0.6$, and correlation coefficient between interest rate and firm value process $\rho = -0.25$. Initial strategic variables are: constant payout rate $\delta = 0.03$, and default threshold $\lambda = 0.6$. To generate payout rate variations, coupon payments are initially set at $c_1 = c_2 = c_3 = 10$ and move together by the same proportions. Credit derivatives are written on the reference obligation with price $P_V(0, c_2, F_2, T_2)$. The maturity date of the default CRO is respectively equal to 2, 4, and 6 years.

the engineering of the CRO maturity brings value added to the basic credit derivative instrument. This is illustrated in the following results.

Table 2 provides the comparative statics for default CRO premia. [7] Interest rate risk and credit risk hedging performances are compared with those of fixed and floating TRS at a QDR equal to 60%.

The default CRO appears as a poor interest rate risk hedging tool (even worse than the fixed TRS for long maturities). Note however that the default CRO with a shorter maturity behaves like a floating TRS with respect to σ_r and α because it promises a discounted (not fixed) payoff. This discount effect also explains that a CRO with a shorter maturity presents a greater sensitivity to interest rate parameters. As expected from Proposition 3, the default CRO with a longer maturity tends to mimic the hedging behaviors of the fixed TRS, except for the interest rate parameters whose impact slightly differs due to the discount factor influencing options premia.

Moreover, the default CRO appears as the most effective credit risk hedging tool. Its American option feature makes it very sensitive to asset volatility (σ), the default threshold (λ) as well as the risk-adjusted dynamics of the state variable (δ). The

[7] The downgrading CRO (studied in the next section) displays qualitatively similar comparative statics. Magnitudes of sensitivities vary with the option strike price but are of comparable order of amount.

sensitivity again increases as option maturity shortens, with the noticeable exception of the recovery rate (Γ) whose importance for option value gets lower, again due to the fact that an early exercise of the option induces the payment of a discounted fraction of the principal amount, whose weight decreases with remaining time to maturity. This finding of a non-monotonic relationship between arrival and recovery rate effects involves that CRO might be a helpful tool to fine-tune credit risk hedging with respect to the relative salience of arrival versus recovery risks of default.

2.2.2. Downgrading CRO

The downgrading CRO entitles its holder the right to obtain a predetermined spread payment on her – still undefaulted – bond when its credit quality reaches a level corresponding to a lowered rating. [8] Upon exercise, the owner still holds the bond but is given a compensation for the degradation in credit standing that resulted in an increase in the yield spread over the remaining life of the bond.

Our modelling approach enables us to consider the total amount of debt for the event of downgrading. Complementary to the mere pricing of the derivative, we are thus not only in the position to track the dynamics of the option value for a given situation, but also to extend the analysis to different leverage structures.

Let $\lambda_1 > \lambda$ be a constant. The associated downgrading time θ_1 may be written as

$$\theta_1 = \inf\{t \geqslant 0 : V_t = \lambda_1 M_t P(t, T^*)\}.$$

More generally, considering N classes of risk, the associated thresholds $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ account for N ratings.

Let cs_0 be the initial promised spot credit spread of the defaultable bond over the riskless bond, i.e. the value of the spot spread solving

$$P_V(0, c, F, T) - FP(0, T)\exp(-\mathrm{cs}_0 T) - c\sum_{i=1}^{n} P(0, t_i)\exp(-\mathrm{cs}_0 t_i) = 0.$$

Similarly, let cs_1 stand for the initial spot credit spread of the lower graded bond over the riskless bond, which solves

$$P_{\lambda_1\Phi}(0, c, F, T) - FP(0, T)\exp(-\mathrm{cs}_1 T) - c\sum_{i=1}^{n} P(0, t_i)\exp(-\mathrm{cs}_1 t_i) = 0,$$

where $P_{\lambda_1\Phi}(0, c, F, T)$ denotes the price of the defaultable bond if the initial value of the state variable is taken at the downgrading threshold. By definition, both cs_0 and cs_1 are known at date 0.

The downgrading CRO has to be viewed as a hedging tool. Thus, this vehicle must ensure at the time of its inception that the effects of a downgrading is completely offset by the resulting payoff from the strike of the option. In order to do so, upon exercise at θ_1, the holder of the option has to get the difference between the remaining cash flows of the bond discounted at rate $r + \mathrm{cs}_0$ and those discounted

[8] Another alternative, followed by Schönbucher (1998), is to consider that the exercise of the CRO leads to an exchange of bonds.

at rate $r + \mathrm{cs}_1$. The pricing of this option is given in the following proposition (see the Appendix A for a proof).

Proposition 5. *In the absence of arbitrage, the value* p_{dwn} *of the downgrading CRO maturing at* $t_{\mathrm{d}} < T$ *satisfies*

$$\begin{aligned} p_{\mathrm{dwn}} = {} & FP(0,T)(\mathrm{e}^{-\mathrm{cs}_0 T}\mathscr{G}_{\mathrm{d}}(\mathrm{cs}_0) - \mathrm{e}^{-\mathrm{cs}_1 T}\mathscr{G}_{\mathrm{d}}(\mathrm{cs}_1)) \\ & + c\sum_{i=l+1}^{n} P(0,t_i)(\mathrm{e}^{-\mathrm{cs}_0 t_i}\mathscr{G}_{\mathrm{d}}(\mathrm{cs}_0) - \mathrm{e}^{-\mathrm{cs}_1 t_i}\mathscr{G}_{\mathrm{d}}(\mathrm{cs}_1)) \\ & + c\sum_{i=1}^{l} P(0,t_i)(\mathrm{e}^{-\mathrm{cs}_0 t_i}\mathscr{G}_{i}(\mathrm{cs}_0) - \mathrm{e}^{-\mathrm{cs}_1 t_i}\mathscr{G}_{i}(\mathrm{cs}_1)), \end{aligned}$$

where

$$\begin{aligned} \mathscr{G}_i(x) = {} & \left(\frac{\lambda_1 \Phi}{V}\right)^{-\frac{1}{2}+\sqrt{\frac{1}{4}+2x}} \mathscr{N}\left(\frac{1}{s(0,t_i)}\left(\ln\frac{\lambda_1\Phi}{V} + \delta t_i + s^2(0,t_i)\sqrt{\frac{1}{4}+2x}\right)\right) \\ & + \left(\frac{\lambda_1 \Phi}{V}\right)^{-\frac{1}{2}-\sqrt{\frac{1}{4}+2x}} \mathscr{N}\left(\frac{1}{s(0,t_i)}\left(\ln\frac{\lambda_1\Phi}{V} + \delta t_i - s^2(0,t_i)\sqrt{\frac{1}{4}+2x}\right)\right). \end{aligned}$$

Fig. 6 plots the structure of downgrading CRO premia for various option maturities and strike prices.

As the CRO is an American option, its value is an increasing function of its maturity regardless of the strike price. In addition, CRO premia are a humped function of the strike price and the hump moves towards a lower strike price as maturity increases. [9]

The presence of a hump is due to two opposite effects. On one hand, a high value of λ_1 moves the downgrading frontier upward, making the event that triggers exercise of the option more likely. On the other hand, the difference in credit spreads shrinks and the option payoff decreases. For a very low level of the threshold, the probability of observing a downgrading of the bond, and thus of striking the option, becomes negligible, although the option payoff is high. By contrast, extremely high values of this threshold go along with bond prices that are very close to the initial one, inducing a very low option payoff.

The hump tends to shift to lower values of the strike when option maturity increases. This means a non-homothetic relationship between the probability and the payoff effects. For low option maturities, where the likelihood of a downgrading event is low, raising the probability of exercise has a major impact on the option value. As CRO maturity increases, the option has greater chance of being exercised and the emphasis naturally shifts to its payoff component. Therefore, the positive

[9] The increased size of the hump with respect to maturity is just the consequence of the maturity effect of the American option.

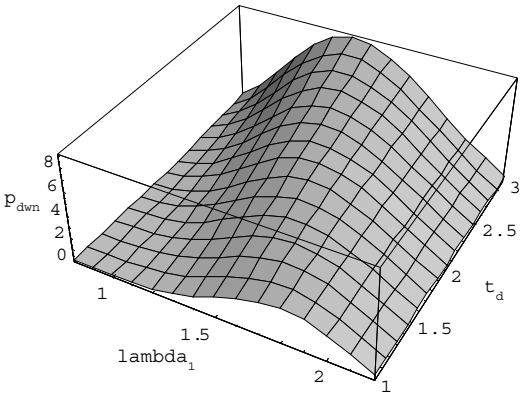

Fig. 6. Downgrading CRO premia for various maturities and strike prices. Interest rate environment is characterized by: mean reversion coefficient $\alpha = 0.1$, long term mean of riskless rate $\beta = 0.06$, current spot rate $r = 0.05$, and interest rate volatility $\sigma_r = 0.02$. Firm characteristics are: current assets value $V = 1000$, principals of debt outstanding: $F_1 = F_2 = F_3 = 200$, coupons are $c_1 = c_2 = c_3 = 10$, debt maturities: $T_1 = 2$, $T_2 = 6$, and $T_3 = 10$, volatility of firm's assets returns $\sigma = 0.2$, recovery rate $\Gamma = 0.6$, default threshold $\lambda = 0.6$, and correlation coefficient between interest rate and firm value process $\rho = -0.25$. The downgrading CRO is written on the reference obligation with price $P_V(0, c_2, F_2, T_2)$ and credit spread $cs_0 = 35$ basis points.

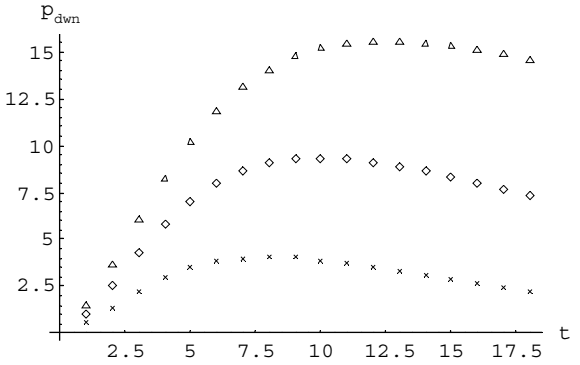

Fig. 7. Term structures of downgrading CRO premia for various coupon payment distributions. Reported coupon payment distributions are $c_1 = 20$, $c_2 = 10$, $c_3 = 0$ (triangles), $c_1 = c_2 = c_3 = 10$ (squares), and $c_1 = 0$, $c_2 = 10$, $c_3 = 20$ (crosses). Interest rate environment is characterized by: mean reversion coefficient $\alpha = 0.1$, long term mean of riskless rate $\beta = 0.06$, current spot rate $r = 0.05$, and interest rate volatility $\sigma_r = 0.02$. Firm characteristics are: current assets value $V = 1000$, principals of debt outstanding: $F_1 = F_2 = 200$, and F_3 is such that the firm QDR is constant at 60%, debt maturities: $T_1 = 2$ and $T_3 = 10$, volatility of firm's assets returns $\sigma = 0.2$, recovery rate $\Gamma = 0.6$, default threshold $\lambda = 0.6$, and correlation coefficient between interest rate and firm value process $\rho = -0.25$. The downgrading CRO is written on the reference obligation with price $P_V(0, c_2, F_2, T_2)$. Its maturity is $t_d = 2$, and its strike price is $\lambda_1 = 1.5$. On the horizontal axis is the number of years of the bond exceeding the CRO maturity, i.e. $t = T - t_d$.

likelihood effect of increasing λ_1 becomes less and less than proportional to the disadvantage of lowering the option payoff.

Finally, Fig. 7 illustrates the impact of terms of debt contracts other than the reference obligation on downgrading CRO premia. Term structures of the downgrading CRO premia are reported for various coupon distributions. In Fig. 7, we have $t_d = 2$ and $\lambda_1 = 1.5$. As in Fig. 4, the curves do not intersect because they are ranked by decreasing levels of Φ, which reduces the downgrading probability.

Term structures of downgrading CRO exhibit a humped shape. Under the credit risky bond pricing model, the behavior of the spread difference $cs_1 - cs_0$ leading the option payoff is an increasing, concave function of bond maturity. For short maturities, the response of this spread difference to a maturity increase is therefore very high, pulling up CRO premia. This explains the first, increasing part of the curve. In the meantime, increasing bond maturity reduces the impact of principal repayment in the option value. Thus, the spread difference increase is gradually offset by the lower discounted value of the principal repayment, which creates the hump for long maturities.

3. Conclusion

Our modeling of bond default risk with a structural form approach departs from a large body of the credit derivatives literature. However, the results proposed in this paper are obtained by relaxing some typical, yet disturbing assumptions of firm value models. We show that the focus on the broader corporate debt structure does not preclude analytical formulations for the value and behavior of the most standard credit derivative contracts.

The pricing and hedging implications originating from this key element allow to emphasize the importance of several dimensions. Even for simple contracts like CDS or TRS, a contract designed on a particular corporate bond experiences very different price sensitivities with respect to the total interest payout, debt principal or corporate leverage. This finding proves to be crucial when considering any change in the liability structure that would leave the QDR unaffected. In particular, the very same phenomenon may lead to drastically opposite results on the credit derivative depending on the maturity of the underlying bond. The ignorance of terms of debt contracts other than the reference obligation would completely offset these issues. Rather, thanks to our integrative approach, such behavior differences outline the clear dependence of credit derivatives prices on broader corporate finance decisions.

The ability to price more complex derivatives in the very same framework provides specific hedging properties associated with each of them. Of remarkable interest is the apparent complementarity of the fixed TRS, floating TRS and default CRO contracts. The first one can be viewed as a hybrid between the other two in terms of interest-rate risk and credit risk hedging properties. However, the option maturity of the default CRO provides an additional degree of freedom whose main consequence is the completely adverse influences of arrival risk and magnitude risk of default. A

further look at term structures of downgrading CRO premia confirms that this type of contract enables to fine-tune hedging properties between default risk components.

Our approach opts for a parsimonious modeling with closed-form expressions for credit derivatives prices and a limited number of parameters to estimate. A possible extension of our work could therefore give up on this analytical tractability in order to better track interest rate dynamics. If a two-factor model of interest rates were to be used, it would certainly be best to use the volatility of the instantaneous risk free rate as the second factor, i.e. σ_r in our model, as in the stochastic volatility term structure (SVTS) model proposed by Fong and Vasicek (1991) where the instantaneous risk free rate variance σ_r^2 follows a positive mean-reverting process. In this realistic setting however, closed-form expression for default probabilities are no longer available. Thus, theoretical resolution of the model would require numerical methods to obtain credit derivatives prices. It is possible nevertheless to figure out some of the effects that a SVTS model would introduce on our results. Indeed, Fong and Vasicek (1991) show the volatility exposure for a zero-coupon bond is an increasing and concave (almost everywhere) function of maturity. Fundamentally, credit derivatives compensate for a possible default loss and their prices behave like the price difference between a riskless bond and its risky counterpart. Fong and Vasicek (1991) show that long-term and short-term riskless and risky bonds are affected by volatility risk in a comparable manner. Hence, the corresponding credit derivatives prices should be little affected by volatility risk. However, for middle-term maturities, the bond price sensitivity to volatility heavily depends on maturity. This is where the duration differential implies a high volatility exposure. Thus, credit derivatives prices may incorporate a non-negligible volatility premium for these maturities.

As a further step for future research, the extent of this contribution has to be examined empirically. In this respect, the paper contains closed-form formulae and term structures analyses that are testable. The implementation of the Fong and Vasicek (1991) model would require the estimation of additional parameters (including the market prices of interest rate and volatility risks), which is certainly a relevant but challenging direction of further empirical investigation.

Appendix A

Proof of Lemma 1. The default boundary may be characterized by

$$H_t = \lambda M_t P(t,T^*) = \lambda\left(\sum_{j=1}^{k(t)} F_j B(T_j,t) + \sum_{j=k(t)+1}^{K} F_j P(t,T_j)\right),$$

which reduces to

$$H_t = \lambda B(0,t)\sum_{j=1}^{K} F_j P(0,T_j).$$

Setting $\Phi = \sum_{j=1}^{K} F_j P(0,T_j)$, the default boundary becomes

$$H_t = \lambda B(0,t)\Phi = \lambda \frac{\Phi}{P(0,T)} P(t,T),$$

which draws back to the model put forward by Briys and de Varenne (1997). □

Proof of Proposition 1. The buyer of a CDS must pay a periodic premium as long as the bond is undefaulted. In return, the counterpart must pay the default loss at date θ, i.e. the present value of the fraction of the coupons and the principal that is not actually paid to bondholders. Holding a long position in the corporate bond and the CDS implies a long position in the equivalent riskless bond. Thus, the current value of the CDS writes

$$\begin{aligned}\mathrm{CDS} = {} & c\sum_{i=1}^{n} P(0,t_i) + FP(0,T) - P_V(0,c,F,T) \\ & - \sum_{i=1}^{n} E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_i} r_u \,\mathrm{d}u\right)\cdot p_{\mathrm{cds}}\cdot 1_{\theta>t_i}\right]\end{aligned}$$

and the solution for p_{cds} is found by setting the contract value equal to 0 at inception. □

Proof of Proposition 2. In the absence of arbitrage, the value to party A of a fixed-for-fixed swap is

$$\mathrm{TRS}_{\mathrm{fix}} = (c - p_{\mathrm{fix}})\sum_{i=1}^{n} P(0,t_i) + FP(0,T) - P_V(0,c,F,T)$$

and is equal to 0 at contract initiation. Similarly, the equation to solve for the fixed-for-floating swap is

$$\begin{aligned}\mathrm{TRS}_{\mathrm{flo}} = {} & \sum_{i=1}^{n} E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_i} r_u \,\mathrm{d}u\right)\cdot Y(t_i,t_{i+1})\cdot F\right] \\ & + FP(0,T) - P_V(0,c,F,T) - p_{\mathrm{flo}}\sum_{i=1}^{n} P(0,t_i),\end{aligned}$$

where $Y(t_i,t_{i+1})$ stands for the yield of the equivalent riskless bond, that is

$$Y(t_i,t_{i+1}) = -\frac{1}{t_{i+1}-t_i}\ln P(t_i,t_{i+1})$$

or, equivalently

$$Y(t_i,t_{i+1}) = \frac{b(t_{i+1}-t_i)r_{t_i} - \ln a(t_{i+1}-t_i)}{t_{i+1}-t_i}.$$

Using the forward neutral measure, we get

$$E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_i} r_u\,\mathrm{d}u\right)\cdot Y(t_i,t_{i+1})\right] = P(0,t_i)E_{\mathbb{Q}_{t_i}}[Y(t_i,t_{i+1})].$$

From the expression of $Y(t_i,t_{i+1})$, we obtain

$$E_{\mathbb{Q}_{t_i}}[Y(t_i,t_{i+1})] = \frac{b(t_{i+1}-t_i)E_{\mathbb{Q}_{t_i}}(rt_i) - \ln a(t_{i+1}-t_i)}{t_{i+1}-t_i}.$$

Under the forward neutral measure, the spot rate mean equals the current instantaneous forward rate. Hence

$$E_{\mathbb{Q}_{t_i}}(r_{t_i}) = (r-\beta)\mathrm{e}^{-\alpha t_i} + \beta - \frac{\sigma_{\mathrm{r}}^2}{2\alpha^2}(1-\mathrm{e}^{-\alpha t_i})^2,$$

which completes the proof. □

Proof of Proposition 3. The discounted expected payoff of the default CRO is

$$\begin{aligned} p_{\mathrm{def}} = {} & F\cdot(1-\lambda\Gamma)\cdot E_{\mathbb{Q}}\left[\exp\left(-\int_0^{T} r_u\,\mathrm{d}u\right)\cdot 1_{\theta<t_{\mathrm{d}}}\right] \\ & + c\cdot\sum_{i=l+1}^{n} E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_i} r_u\,\mathrm{d}u\right)\cdot 1_{\theta<t_{\mathrm{d}}}\right] \\ & + c\cdot\sum_{i=1}^{l} E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_i} r_u\,\mathrm{d}u\right)\cdot 1_{\theta<t_i}\right], \end{aligned}$$

where $l = \sup\{i \geqslant 0 : t_i \leqslant t_{\mathrm{d}}\}$. The third expectation follows from observing that

$$E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_i} r_u\,\mathrm{d}u\right)\cdot 1_{\theta<t_i}\right] = P(0,t_i) - E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_i} r_u\,\mathrm{d}u\right)\cdot 1_{\theta>t_i}\right].$$

The first two expectations can be rewritten, observing that, for any $t > t_{\mathrm{d}}$

$$\begin{aligned} & E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t} r_u\,\mathrm{d}u\right)\cdot 1_{\theta<t_{\mathrm{d}}}\right] \\ & \quad = E_{\mathbb{Q}}\left[\exp\left(-\int_{t_{\mathrm{d}}}^{t} r_u\,\mathrm{d}u\right)\cdot\exp\left(-\int_0^{t_{\mathrm{d}}} r_u\,\mathrm{d}u\right)\cdot 1_{\theta<t_{\mathrm{d}}}\right] = P_{\mathrm{f}}(t_{\mathrm{d}},t)\cdot x_{\mathrm{d}}, \end{aligned}$$

where

$$P_{\mathrm{f}}(t_{\mathrm{d}},t) = \exp\left[-\int_{t_{\mathrm{d}}}^{t}\left((r-\beta)\mathrm{e}^{-\alpha u} + \beta - \frac{\sigma_{\mathrm{r}}^2}{2\alpha^2}(1-\mathrm{e}^{-\alpha u})^2\right)\mathrm{d}u\right]$$

is the forward price of a bond at time t_{d} and maturing at t. □

Proof of Proposition 4. Setting

$$\Lambda(\theta_1,t) = \exp\left(-\int_{\theta_1}^{t}(r_u+\mathrm{cs}_0)\,\mathrm{d}u\right) - \exp\left(-\int_{\theta_1}^{t}(r_u+\mathrm{cs}_1)\,\mathrm{d}u\right),$$

we have that

$$p_{\text{dwn}} = FE_{\mathbb{Q}}\left[\exp\left(-\int_0^{\theta_1} r_u\,\mathrm{d}u\right)\cdot 1_{\theta_1\leqslant t_{\text{d}}}\cdot \Lambda(\theta_1,T)\right] + c\sum_{i=l+1}^{n} E_{\mathbb{Q}}\left[\exp\left(-\int_0^{\theta_1} r_u\,\mathrm{d}u\right)\cdot 1_{\theta_1\leqslant t_{\text{d}}}\cdot \Lambda(\theta_1,t_i)\right] + c\sum_{i=1}^{l} E_{\mathbb{Q}}\left[\exp\left(-\int_0^{\theta_1} r_u\,\mathrm{d}u\right)\cdot 1_{\theta_1\leqslant t_i}\cdot \Lambda(\theta_1,t_i)\right],$$

which yields

$$p_{\text{dwn}} = FE_{\mathbb{Q}}\left[\exp\left(-\int_0^{T} r_u\,\mathrm{d}u\right)\cdot 1_{\theta_1\leqslant t_{\text{d}}}\cdot \left(\mathrm{e}^{-\mathrm{cs}_0(T-\theta_1)}-\mathrm{e}^{-\mathrm{cs}_1(T-\theta_1)}\right)\right] + c\sum_{i=l+1}^{n} E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_i} r_u\,\mathrm{d}u\right)\cdot 1_{\theta_1\leqslant t_{\text{d}}}\cdot \left(\mathrm{e}^{-\mathrm{cs}_0(t_i-\theta_1)}-\mathrm{e}^{-\mathrm{cs}_1(t_i-\theta_1)}\right)\right] + c\sum_{i=1}^{l} E_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_i} r_u\,\mathrm{d}u\right)\cdot 1_{\theta_1\leqslant t_i}\cdot \left(\mathrm{e}^{-\mathrm{cs}_0(t_i-\theta_1)}-\mathrm{e}^{-\mathrm{cs}_1(t_i-\theta_1)}\right)\right].$$

Applying the forward neutral risk measure, this formula easily reduces to

$$p_{\text{dwn}} = FP_{\text{f}}(t_{\text{d}},T)P(0,t_{\text{d}})\left[\mathrm{e}^{-\mathrm{cs}_0 T}E_{\mathbb{Q}_{t_{\text{d}}}}(1_{\theta_1\leqslant t_{\text{d}}}\cdot \mathrm{e}^{\mathrm{cs}_0\theta_1})-\mathrm{e}^{-\mathrm{cs}_1 T}E_{\mathbb{Q}_{t_{\text{d}}}}(1_{\theta_1\leqslant t_{\text{d}}}\cdot \mathrm{e}^{\mathrm{cs}_1\theta_1})\right] + c\sum_{i=l+1}^{n} P_{\text{f}}(t_{\text{d}},t_i)P(0,t_{\text{d}})\left[\mathrm{e}^{-\mathrm{cs}_0 t_i}E_{\mathbb{Q}_{t_{\text{d}}}}(1_{\theta_1\leqslant t_{\text{d}}}\cdot \mathrm{e}^{\mathrm{cs}_0\theta_1})-\mathrm{e}^{-\mathrm{cs}_1 t_i}E_{\mathbb{Q}_{t_{\text{d}}}}(1_{\theta_1\leqslant t_{\text{d}}}\cdot \mathrm{e}^{\mathrm{cs}_1\theta_1})\right] + c\sum_{i=1}^{l} P(0,t_i)\left[\mathrm{e}^{-\mathrm{cs}_0 t_i}E_{\mathbb{Q}_{t_i}}(1_{\theta_1\leqslant t_i}\cdot \mathrm{e}^{\mathrm{cs}_0\theta_1})-\mathrm{e}^{-\mathrm{cs}_1 t_i}E_{\mathbb{Q}_{t_i}}(1_{\theta_1\leqslant t_i}\cdot \mathrm{e}^{\mathrm{cs}_1\theta_1})\right].$$

And the desired result follows from noting that, using standard computation techniques,

$$\mathscr{G}_i(x) = E_{\mathbb{Q}_{t_i}}(1_{\theta_1\leqslant t_i}\cdot \mathrm{e}^{x\theta_1}) = \left(\frac{\lambda_1 FP(0,t_i)}{V}\right)^{-\frac{1}{2}+\sqrt{\frac{1}{4}+2x}}\mathscr{N}\left(\frac{\ln\frac{\lambda_1 FP(0,t_i)}{V}}{s(0,t_i)}+s(0,t_i)\sqrt{\frac{1}{4}+2x}\right) + \left(\frac{\lambda_1 FP(0,t_i)}{V}\right)^{-\frac{1}{2}-\sqrt{\frac{1}{4}+2x}}\mathscr{N}\left(\frac{\ln\frac{\lambda_1 FP(0,t_i)}{V}}{s(0,t_i)}-s(0,t_i)\sqrt{\frac{1}{4}+2x}\right). \qquad \square$$

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